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AN EXPANSION FORMULA FOR THE I-FUNCTIONS OF SEVERAL VARIABLES

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Abstract

In this paper an expansion formula for the I-function of several variables has been obtained. Many interesting new results can be obtained by specializing the parameters of the I-functions of several variables.

1. Introduction

Notations and Results used :

$(a)_n$ stands for $a(a+1)\cdots(a+n-1)$

${}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},A_j)_p$ stands for $(a_1;\alpha_1^{(1)},\cdots,\alpha_1^{(r)},A_1),(a_2;\alpha_2^{(1)},$
$\cdots,\alpha_2^{(r)};A_2),\cdots,(a_p;\alpha_p^{(1)},\cdots,\alpha_p^{(r)};A_p)$

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$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad n \geq 1. \tag{1.1}$$

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [6, p.38] is defined and represented as:

$$I[z_1, \cdots, z_r] = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}$$

$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p :_1 (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_q :_1 (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;_1 (d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r) z_1^{s_1}\cdots z_r^{s_r} ds_1 \cdots ds_r, \tag{1.2}$$

where $\phi(s_1, \cdots, s_r)$ and $\theta_i(s_i), i = 1, 2, \cdots, r$ are given by,

$$\varphi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n} \Gamma^{A_j}\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i\right)}{\prod_{j=1}^{q} \Gamma^{B_j}\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i\right) \prod_{j=n+1}^{p} \Gamma^{A_j}\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i\right)}, \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}}(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}}(1 - c_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}}(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}}(c_j^{(i)} - \gamma_j^{(i)} s_i)}, \tag{1.4}$$

Also $z_i \neq 0$ $(i = 1, \cdots, r)$, $\omega = \sqrt{-1}$, m_j, n_j, p_j, q_j $(j = 1, \cdots, r)$, n, p, q are non-negative integers such that $0 \leq n \leq p, q \geq 0, 0 \leq m_j \leq q_j$, $0 \leq n_j \leq p_j$ $(j = 1, 2, \cdots, r)$ (not all zero simultaneously).

$\alpha_j^{(i)}$ $(j = 1, 2, \cdots, p, i = 1, 2, \cdots, r)$, $\beta_j^{(i)}$ $(j = 1, 2, \cdots, q, i = 1, 2, \cdots, r)$, $\gamma_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$, and $\delta_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ are positive numbers.

a_j $(j = 1, 2, \cdots, p)$, b_j $(j = 1, 2, \cdots, q)$, $c_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$, and $d_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ are complex numbers. The exponents A_j $(j = 1, 2, \cdots, p)$, B_j $(j = 1, 2, \cdots, q)$, $C_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$ and $D_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ of various gamma functions may take non integer values. The I-functiion of r-variables is analytic if

$$\Psi_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0, \quad i = 1, 2, \cdots, r.$$

The integral (1.2) converges absolutely if $|(arg(z_i)| < \frac{1}{2}\Delta_i\pi, i = 1, 2, \cdots, r$ where

$$\begin{aligned}\Delta_i &= -\sum_{j=n+1}^{p} A_j\alpha_j^{(i)} - \sum_{j=1}^{q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)} \\ &\quad + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{i} > 0.\end{aligned} \tag{1.5}$$

For further details refer [6].

Mathai and Saxena [4, p. 62]

$$\sqrt{\pi}\frac{\Gamma(s+1)}{\Gamma(s+1/2)}\left(\cos\frac{\theta}{2}\right)^{2s} = 1 + 2\sum_{k=1}^{\infty}\frac{(-1)^k(-s)_k}{(s+1)_k}\cos(k\theta), \tag{1.6}$$

where $R(s) > -\frac{1}{2}, \quad 0 \leq \theta \leq \pi$.

2. Main Result

$$\begin{aligned}&\sqrt{\pi} I[z_1(1+\cos\theta)^{\alpha_1}, \cdots, z_r(1+\cos\theta)^{\alpha_r}] \\ &= I^{0,n+1:m_1,n_1;\cdots;m_r,n_r}_{p+1,q+1:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|c} 2^{\alpha_1}z_1 & I_1 \\ \vdots & \\ 2^{\alpha_r}z_r & I_2\end{array}\right] + 2\sum_{k=1}^{\infty}\cos(k\theta) \\ &I^{0,n+2:m_1,n_1;\cdots;m_r,n_r}_{p+2,q+2:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|c} 2^{\alpha_1}z_1 & I_3 \\ \vdots & \\ 2^{\alpha_r}z_r & I_4\end{array}\right], \quad (2.1)\end{aligned}$$

where,

$$\begin{aligned}I_1 &= \left(\frac{1}{2};\alpha_1,\cdots,\alpha_r;1\right),\ {}_1(\alpha_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p :\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ I_2 &= {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_q,(0;\alpha_1,\alpha_2,\cdots,\alpha_r;1) :\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots,\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}, \\ I_3 &= \left(\frac{1}{2};\alpha_1,\cdots,\alpha_r;1\right),(0;\alpha_1,\cdots,\alpha_r;1),\ {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p : \\ &\quad {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}, \\ I_4 &= {}_1(b_j;\beta_j^{(1)},\cdots,b_j^{(r)};B_j)_q,(-k;\alpha_1,\alpha_2,\cdots,\alpha_r;1),(k;\alpha_1,\alpha_2,\cdots,\alpha_r;1) : \\ &\quad {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}.\end{aligned}$$

Provided

(i) $0 \leq \theta \leq \pi, \quad \alpha_k > 0,$

(ii) $Re\left(\frac{1}{2} + \sum_{i=1}^{r} \frac{\alpha_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, j = 1, 2, \cdots, m_i,$

(iii) $arg(z_k) < \frac{1}{2}\Delta_k \pi, k = 1, 2, \cdots, r,$

where Δ_k is given by (1.5).

Proof : Expressing the I-function of r-variables on the left hand side of (2.1) as a contour integral using (1.2), the left side of (2.1) becomes:

$$\sqrt{\pi}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\varphi(s_1,\cdots,s_r) \times z_1^{s_1}\cdots z_r^{s_r}2^{\alpha_1 s_1+\cdots+\alpha_r s_r}\left(\cos\frac{\theta}{2}\right)^{2(\alpha_1 s_1+\cdots+\alpha_r s_r)}ds_1\cdots ds_r, \tag{2.2}$$

where $\phi(s_1, \cdots, s_r)$ and $\theta_i(s_i), i = 1, 2, \cdots, r$ are given by,

$$\varphi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)},$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}.$$

Using (1.6), the equation (2.2) reduces to:

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\varphi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}2^{\alpha_1 s_1+\cdots+\alpha_r s_r} \times\left[\frac{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1/2)}{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1)}+2\sum_{k=1}^{\infty}\frac{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1)\ \Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1/2)}{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1-k)\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1+k)}\cos(k\theta)\right]ds_1\cdots ds_r. \tag{2.3}$$

Now changing the order of integration and summation which is justified under the given conditions, (2.3) becomes:

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\varphi(s_1,\cdots,s_r)(2^{\alpha_1}z_1)^{s_1}\cdots(2^{\alpha_r}z_r)^{s_r}\frac{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+\frac{1}{2})}{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1)}ds_1\cdots ds_r$$
$$+2\sum_{k=1}^{\infty}\cos(k\theta)\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\varphi(s_1,\cdots,s_r)(2^{\alpha_1}z_1)^{s_1}\cdots(2^{\alpha_r}z_r)^{s_r}$$
$$\times\frac{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1)\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1/2)}{\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1-k)\Gamma(\alpha_1 s_1+\cdots+\alpha_r s_r+1+k)}ds_1\cdots ds_r. \tag{2.4}$$

Now interpret this integral (2.4) with the help of (1.2), we get the required result.

3. Special Cases

When $r = 2$, (2.1) reduces to:

$$\sqrt{\pi} I[z_1(1+\cos\theta)^{\alpha_1}, z_2(1+\cos\theta)^{\alpha_2}] = I^{0,n+1:m_1,n_1;m_2,n_2}_{p+1,q+1:p_1,q_1;p_2,q_2}$$

$$\left[\begin{array}{c|c} 2^{\alpha_1} z_1 & \left(\frac{1}{2};\alpha_1,\alpha_2;1\right),\ {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_p:\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1},\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ 2^{\alpha_2} z_2 & {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_q,(0;\alpha_1,\alpha_2;1):\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array}\right]$$

$$2\sum_{k=2}^{\infty}\cos(k\theta) I^{0,n+2:m_1,n_1;m_2,n_2}_{p+2,q+2:p_1,q_1;p_2,q_2}$$

$$\left[\begin{array}{c|c} 2^{\alpha_1} z_1 & \left(\frac{1}{2};\alpha_1,\alpha_2;1\right),(0;\alpha_1,\alpha_2;1)\ {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_p:\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1},\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ \vdots & \\ 2^{\alpha_2} z_2 & {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_q,(-k;\alpha_1,\alpha_2;1)(k,\alpha_1,\alpha_2;1):\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array}\right] \tag{3.1}$$

provided the conditions are similar to that of (2.1) with $r = 2$.
When $n = p = q = 0$ and $r = 1$ and specializing the parameters of (2.1) , (2.1) reduces to the following Corollary for I- function of one variable.
Corollary :

$$\begin{aligned} \sqrt{\pi} I[z(1+\cos\theta)^{\alpha}] &= I^{m,n+1}_{p+1,q+1}\left[2^{\alpha}z \left| \begin{array}{c} \left(\frac{1}{2};\alpha;1\right),\ {}_1(a_j;\alpha_j;A_j)_p \\ {}_1(b_j;\beta_j;B_j)_q,(0;\alpha;1) \end{array}\right.\right] \\ &+2\sum_{k=1}^{\infty}\cos(k\theta) I^{m,n+2}_{p+2,q+2}\left[2^{\alpha}z \left| \begin{array}{c} \left(\frac{1}{2};\alpha;1\right),(0;\alpha;1)\ {}_1(a_j;\alpha_j;A_j)_p \\ {}_1(b_j;\beta_j;B_j)_q,(-k;\alpha;1),(k;\alpha;1) \end{array}\right.\right] \end{aligned} \tag{3.2}$$

When $A_j = B_j = 1$ and z is replaced by $\frac{z}{2^{\alpha}}$, (3.2) reduces to a result given by Parashar [5].

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